SELF-SIMILAR SOLUTION OF THE EQUATIONS OF NONLINEAR DIFFUSION OF A MAGNETIC FIELD

V. V. Prut

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The equations, boundary conditions, and initial conditions which describe the model of the diffusion of a magnetic field in space (an incompressible, current-conducting but not heat-conducting medium) can be written in the form

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\varkappa\frac{\partial H}{\partial x},\quad \frac{\partial Q}{\partial t} = \frac{\varkappa}{4\pi}\left(\frac{\partial H}{\partial x}\right)^2, \tag{1}$$

$$H(x=0,\ t) \equiv H_0 = at^{\alpha},\ H(x=\infty,\ t) = 0,$$

$$H(x,\ t=0) = 0,\ Q(x,\ t=0) = Q_0,$$

where $\varkappa = c^2/4\pi\sigma\mu$ is the diffusion coefficient, $[Q] = \mathrm{erg/cm^3}$, the remaining notation is that generally employed, and the system of units is Gaussian.

Some results on a self-similar solution of the equations of the diffusion of a magnetic field are given in [1-4]. Unlike [1-4], we will assume that $\varkappa = bQ^{\beta}$, where b and β are constants, defined as a certain mean of the experimental or theoretical dependences of the electrical resistance on temperature or enthalpy. For the majority of metals this relationship is linear (β = 1) when $T_D \leqslant T \leqslant T_m$, where T_D and T_m are the Debye temperature and the melting point; over a wider range β > 1: In particular when $T \leqslant 0.2\ T_D$, $\beta = 5/4$ ($\varkappa \sim T^5$, and $Q \sim T^4$), and When $T > T_m$, β > 1 due to the jump in resistance when $T = T_m$; for certain alloys $\beta \approx 0$, for high melting point metals $\beta \leqslant 1$, and for plasma β = −3/2. Here we manly consider values of $B \geqslant 0$, and also μ = 1.

Problem (1) is defined by three constants with independent dimensions:

$$[a] = L^{-1/2}M^{1/2}T^{-1-\alpha},\quad [b] = L^{2+\beta}M^{-\beta}T^{2\beta-1},\ [Q_0] = L^{-1}M^{1}T^{-2}.$$

If we take these as the basic units of measurement, the equations are completely dimensionless (for this we can assume in all the equations (1) that a, b, and Q_0 are unity. Hence, it is obvious that for specified α and β it is sufficient to solve the dimensionless equations completely in partial derivatives only once in all.

In intense magnetic fields, in the region close to the boundary of the half-space, the Joule energy $Q \gg Q_0$, and the material "forgets" its initial state, and problem (1) is defined solely by two parameters, namely a and b. (When β = 0, the problem is always defined essentially by only the parameters a and b, since the energy is measured from an arbitrary initial energy.) In this case, as is well known (see, for example, [5, 6]), the problem is self-similar. We will obtain a solution of the more general problem (1) by considering the self-similar solution as the limiting case as $t \to \infty$ or $Q_0 \to 0$.

We will replace the variables and the functions in such a way that this replacement remains true even when $Q_0 = 0$. Then, Q_0 only affects the time taken to transfer to the self-similar solution and the boundary of the region where it is applicable. It can be shown that the self-similar variable can be represented in the form

$$\xi = x/dt^{\delta},\ \text{where}\ \delta = 1/2 + \alpha\beta,\ d = (a/\sqrt{8\pi})^{\beta}b^{1/2}.$$

We will replace the functions as follows: $H = at^{\alpha}h(\xi,\ t)$, $Q = (a^2t^{2\alpha}/8\pi) \times q(\xi,\ t) + Q_0$, in which case we have

$$j = -\frac{c}{4\pi}\frac{\partial H}{\partial x} = \frac{c}{4\pi}\frac{a}{d}t^{\alpha-\delta}i,\ \text{where}\ i = -\partial h/\partial\xi;$$

h, q, and i are the "self-similar" magnetic field, energy, and current density.

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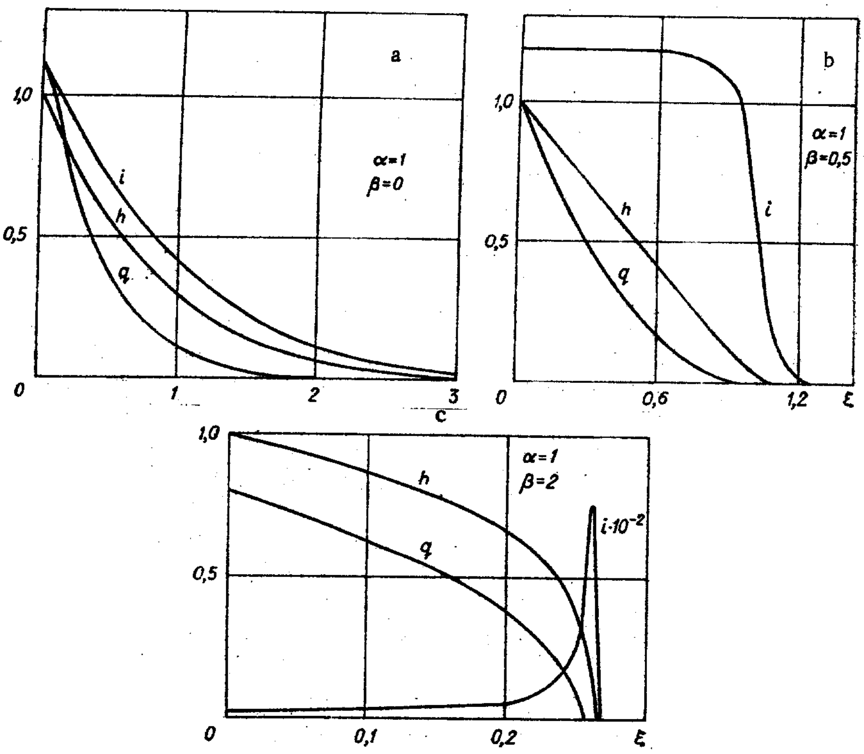

Fig. 1

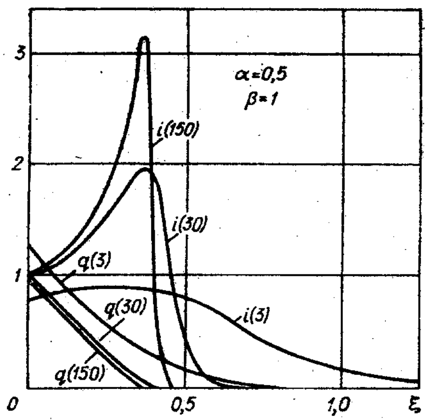

Fig. 2

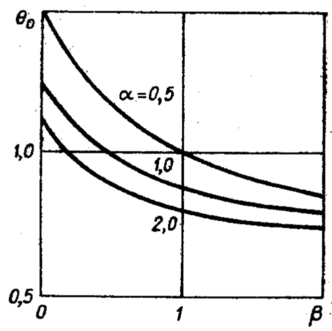

Fig. 3

By making these replacements in Eqs. (1) we obtain

$$\frac{\partial h}{\partial \ln t} = \frac{\partial}{\partial \xi}\chi\frac{\partial h}{\partial \xi} + \delta\xi\frac{\partial h}{\partial \xi} - \alpha h,$$
$$\frac{\partial q}{\partial \ln t} = 2\chi\left(\frac{\partial h}{\partial \xi}\right)^2 + \delta\xi\frac{\partial q}{\partial \xi} - 2\alpha q, \qquad (2)$$

where $\chi = [q + Q_0/H_0^2/8\pi)]^{\beta}$; $h(\xi = 0, t) = 1$; $h(\xi = \infty, t) = 0$.

If $Q_0/H_0^2/8\pi) \ll q$, the coefficients on the right sides of (2) will be independent of time, and the equations will reduce to the self-similar form. However, the solution of the boundary value problem for nonlinear ordinary differential equations is a problem which is, perhaps, more nontrivial than the solution of a nonlinear equation in partial derivatives. Moreover, one of the methods of solving ordinary equations is the establishment method, which is similar in concept to a nonstationary problem. Hence, it is more convenient to solve the nonself-similar equations (2) by assuming that for large t the solution is close to the self-similar solution.

To solve Eqs. (2) when $\delta > 0$ we will use the following diference scheme:

$$\frac{h_n^{m+1}-h_n^m}{\Delta_m}=\frac{-1}{\Delta_n}\left(\chi_{n+1/2}^m i_{n+1/2}^{m+1}-\chi_{n-1/2}^m i_{n-1/2}^{m+1}\right)-\delta\xi_n i_{n+1/2}^{m+1}-\alpha h_n^{m+1},$$
$$\frac{q_{n+1/2}^{m+1}-q_{n+1/2}^m}{\Delta_m}=2\chi_{n+1/2}^m\left(i_{n+1/2}^{m+1}\right)^2+\delta\xi_{n+1/2}\frac{q_{n+3/2}^{m+1}-q_{n+1/2}^{m+1}}{\Delta_{n+1}}-2\alpha q_{n+1/2}^{m+1}, \tag{3}$$

where $i_{n+1/2} = -(h_{n+1}-h_n)/\Delta_{n+1/2}$; Δ_m and Δ_n are the steps and m and n are the indices of ln t and ξ.

Unlike (1), a feature of Eqs. (2) is the presence in it of the first derivatives with respect to ξ, responsible, in this system of coordinates, for the transfer of the quantities h and q. These terms were approximated using a scheme of first-order accuracy. The equation of the field diffusion in (3) was solved by the method of pivotal condensation, while the energy was found explicitly: assuming q to be from right to left. For small t we solved Eq. (1) numerically and then, for t defined by the condition $Q_0/(H_0^2/8\pi) \sim 1$, we made the transition to Eqs. (2).

The results of the solution for $\alpha = 1$ and $\beta = 0$, 0.5, and 2 in the form of curves of h, i, and q against ξ with $\tau = 30$ (τ is the time in the system of units $\{a, b, Q_0\}$) are shown in Fig. 1a–c, illustrating the main behavior when the nonlinearity, i.e., β, increases. For linear diffusion, as is well known (see, for example, [1]), the magnetic field within the framework of the model described by Eq. (1) propagates instantaneously over all space, although with a very sharp (Gaussian) fall-off of the field, so that the main portion of the field energy propagates as $x \sim t^{1/2}$.

For nonlinear diffusion, i.e., when $\beta \neq 0$, and in addition when $Q_0 = 0$, a clear boundary occurs which separates the region with the field from the region without the field. The field and the field energy and internal energy respectively propagate from the boundary of the half-space in the form of a wave the front of which has a noninfinite velocity. This process is very similar to the propagation of a thermal wave [5], so that it is natural to call this a magnetic wave.

The distribution of the field and the internal energy over the front can be found if we seek a solution of (1) in the form $H = H(s)$, $Q = Q(s)$, where $s = x - v_f$ and $v_f = \text{const}$. Equations (1) take the form

$$-v_\Phi\frac{\partial H}{\partial s}=\frac{\partial}{\partial s}bQ^\beta\frac{\partial H}{\partial s},\quad -v_\Phi\frac{\partial Q}{\partial s}=bQ^\beta\left(\frac{\partial H}{\partial s}\right)^2. \tag{4}$$

The solution of (4) will be sought in the form $H \sim s^\lambda$ with an undefined coefficient λ, but substituting which into (4) we obtain $\lambda = 1/2\beta$ and correspondingly $H \sim (x_f - x)^{1/2\beta}$, $Q \sim (x_f - x)^{1/\beta}$. The current density on the front $j \sim (x_f - x)^{1/2\beta-1}$, so that when $\beta < 1/2 j_f = 0$, when $\beta > 1/2 j_f = \infty$ and when $\beta = 1/2 j_f = \text{const} \neq 0$; the gradient of the internal energy $\partial Q/\partial x \sim (x_f - x)^{1/\beta-1}$, whence on the front $\partial Q/\partial x = 0$ when $\beta < 1$, etc. The numerical results in Fig. 1 agree qualitatively with these simple estimates.

The velocity of the front, generally speaking, is not constant, and is found from the self-similar variable $v_f = dx_f/dt = \xi_f d\delta t^{\delta-1}$, so that when $\delta > 0$ the wave propagates into the depth of the conductor, when $\delta = 0$ the wave is a standing wave, and when $\delta < 0$ the wave travels to the boundary of the half-space (this is only possible when $\beta < 0$). The velocity is constant only when $\delta = 1$; when $\delta < 1$ the velocity decreases from infinity to zero, and when $\delta < 1$ it increases. Note that for linear diffusion with $\delta = 1/2$ the velocity of the front, as is well known, decreases, and for nonlinear diffusion with $\alpha = 1$, $\beta = 1$, and $\delta = 3/2$, the velocity increases. The coordinate of the front ξ_f in the self-similar variables can be found from the solution of Eqs. (3) and correspondingly $x_f = \xi_f d t^\delta$. When $H_0/\sqrt{8\pi Q_0} \gg 1$, the nonlinearity increases the region of space occupied by the intense field. The current density — physical and not self-similar — is determined by the time-dependence as $j \sim t^{-[1/2+\alpha(\beta-1)]}$, so that it may decrease or increase (when $\alpha(1-\beta) > 1/2$).

The finiteness of the velocity of the front when $\beta \neq 0$ and $Q_0 = 0$ is due to the fact that the diffusion coefficient is the coefficient of the leading derivative $\varkappa = 0$ and the differential parabolic equation is degenerate. When $\beta \neq 0$ and $Q_0 \neq 0$, $\varkappa \neq 0$ and the velocity of the front becomes infinite, but in intense fields when $Q \gg Q_0$ the "self-similar" processes

TABLE 1

	Ag	Al	Cu	Fe	Ta	W
Q_c, 10^{11} erg/cm^3	2,7	3,2	4,7	5,4	7,1	8,8
H_c, MOe	2,6	2,8	3,5	3,7	4,2	4,7

described above play the main role in the energy transfer processes. The effect of $Q_0 \neq 0$ reduces to spreading of the front and to its instantaneous propagation. The main fraction of the energy is transferred with a velocity defined by the self-similar variable (in which Q_0 does not occur) while a small fraction lies between the self-similar front and infinity.

The characteristic transfer time t_a to the self-similar solution is determined from $q \sim Q_0/(H_0^2/8\pi)$, so that when $q \sim 1$, $t_a \sim (8\pi Q_0/a^2)^{1/2\alpha}$ or $\tau \sim (8\pi)^{1/2\alpha}$. The relations $q(\xi)$ and $i(\xi)$ in Fig. 2 illustrate the self-similarity of the process, where we show in brackets the corresponding instants of time τ. The characteristic value of the blurring of the front is found from (1); if the solution is sought in the form $H = H(x - v_f t)$, then $H \sim \exp(-(x - x_f)/\Delta)$, where $\Delta = \varkappa_0/v_f$.

From the practical point of view the relation between the internal and the magnetic energies is of most interest. We will introduce the coefficient θ found from $Q(x, t) = \theta H^2(x, t)/8\pi$, so that in the region of self-similarity $\theta = q(\xi)/h^2(\xi)$. For certain values of α the dependences of $\theta_0 \equiv \theta(\xi = 0) = q$ on β are shown in Fig. 3 (with $\tau = 5/0.03^{1/2\alpha}$). When β increases, irrespective of α, θ_0 falls slowly, and when $\alpha \sim 1$, $\theta_0 \sim 1$. The latter result is natural since there is no small parameter in the problem and there is one variable with the dimensions of energy. As $\alpha \to 0$, $\theta_0 \to \infty$, analogous to the situation observed for linear diffusion [1] and nonlinear diffusion with $\beta \approx 1$ and $\alpha = 0$ [2], i.e., an increase in the rate of growth of the field leads to an increase in j and correspondingly θ.

Loss of conductivity occurs when $\rho \sim \rho_c$ (ρ_c is the critical density), where the characteristic value of the energy is the binding energy Q_c. From the condition $\theta \sim 1$, it follows that a certain characteristic quantity of the maximum magnetic field $H_c = (8\pi Q_c)^{1/2}$ exists. Q_c and H_c are given for certain elements in Table 1. This field may be exceeded appreciably for several reasons. First, due to the intertia of the scattering of the conductor; in this case, the condition $t_i < t$ must be satisfied, where the inertia time $t_i \sim x/c$, x is the value of the front or the thickness of the conductor, and c is the velocity of sound. If $x = x_f$, $c \sim T^{1/2} \sim t^{\alpha}$, this condition reduces to $t^{\alpha\beta - 1/2 - \alpha} <$ const, characteristic for a conductor.

Second, the magnetic pressure $p_H = H^2/8\pi$ may exceed the thermodynamic pressure p of the material. We will assume that $p = p_X + p_T$, where p_X is the elastic (potential) pressure and $p_T + \gamma Q_T$ is the thermal pressure; Q_X is the potential energy and $Q_T = Q - Q_X$ is the thermal energy. When $p_X \approx 0$ or $p_T \gg p_X$ $p = \gamma(\theta p_H - Q_X)$ and when $\theta p_H \gg Q_c$ and $\gamma\theta < 1$ the magnetic pressure prevents dispersion of the conductor, the conductivity of which must already be high. If the material expands so much that it becomes a dense plasma and β changes sign, it is natural qualitatively that as $T \to \infty$ the value of the skin layer and the total energy dissipitated approach zero. A reduction in θ and a change in the relations on the wavefront become possible, in particular, due to the thermal conductivity and the radiation.

On the other hand, under practical conditions the maximum value of the field may be limited for technical reasons or due to hydrodynamic effects or instabilities not considered here.

The limitation of the "thickness" R of the half-space with respect to the wavefront x_f leads to an increase in θ. This follows from the fact that the mean current density $j \sim H/R$ increases, and correspondingly $Q \sim \int \varkappa j^2 dt$. We will consider the limiting case which widely occurs when a wire explodes, when the current is uniformly distributed over the thickness. Then

$$j \sim \frac{\partial H}{\partial x} \sim \frac{H_0}{R} = \frac{at^{\alpha}}{R}, \quad \frac{\partial Q}{\partial t} = bQ^{\beta} \frac{a^2}{4\pi} \frac{t^{2\alpha}}{R^2};$$

integrating this equation we obtain that when $\beta = 1$ $Q = Q_0 \exp(\eta)$ and when $\beta \neq 1$

$$Q = [Q_0^{1-\beta} + \eta(1-\beta)]^{1/(1-\beta)}, \text{ where } \eta = \frac{ba^2}{4\pi} \frac{t^{2\alpha+1}}{2\alpha+1} \frac{1}{R^2};$$

in particular, when $\beta < 1$ the relation $Q = Q(t)$ is approximately a power relationship, when $\beta = 1$ it is exponential, and when $\beta > 1$ it is hyperbolic, and as one approaches the corresponding asymptote the whole conductor explodes.

In conclusion we note that these results are also applicable to the following cases. First, when the field is derived from a conductor: We can consider a model in which the initial field $H(x) = \text{const}$, the field is measured from this value, and we use $H = -at^{\alpha}$ as the boundary condition. Second, when the heating occurs at constant pressure, we can assume that Q is the enthalpy and x is a coordinate which moves with the material.

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BREAKDOWN VOLTAGES OF INERT GASES AT TEMPERATURES OF 300-2000°K

I. V. Bozhko, N. I. Glazkov,
S. R. Troitskii, and N. I. Fal'kovskii

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We may assume that as temperature increases, the dielectric strength of inert gases begins to deviate from Paschen's law

$$U_{\text{br}} = f(ps/T) \tag{1}$$

at lower temperatures than, for example, the dielectric strength of electronegative gases [1]. This may be due, in particular, to the fact that in the inert gas there is no capture of the thermal electrons emitted by the cathode at high temperatures, i.e., there is no factor which will retard the development of breakdown. A confirmation of this may be found in the results of an investigation of currents in unheated inert gases at pressures p < 40 kPa [2], which disclosed the phenomenon of early breakdown, attributed by the authors to the thermal ionization of the gas near the cathode filament, which is incandescent above 2400°K.

George and Messerle [3] obtained breakdown voltages for argon and helium at isothermal conditions, T = 1600-2500°K, but in a nonuniform field, and therefore no generalized conclusions can be drawn from their results. Measurement of the dielectric strength of argon in a shock tube [4] showed that the breakdown of the gas under such conditions is determined by the colder boundary layer; this makes interpretation of the results more difficult. Direct measurements of the pulsed dielectric strength when a stream of gas is heated in a plasmotron were carried out in [5]. The results show that up to 2100°K in argon and helium Paschen's law remains valid in investigations using unaged electrodes, while investigations with aged electrodes yield much higher values than Paschen's law. Since no conclusion con-

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